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PRERESOLUTIONS OF NONCOMMUTATIVE ISOLATED SINGULARITIES

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We introduce the notion of right preresolutions (quasiresolutions) for noncommutative isolated singularities, which is a weaker version of quasiresolutions introduced by Qin, Wang and Zhang (*J. Algebra* 536 (2019), 102–148). We prove that right quasiresolutions for a noetherian bounded below and locally finite graded algebra with right injective dimension 2 are always Morita equivalent. When we restrict to a noncommutative quadric hypersurface A, we prove that if A is a noncommutative isolated singularity, then it always admits a right preresolution. We provide a method to verify whether a noncommutative quadric hypersurface is an isolated singularity. An example of noncommutative quadric hypersurfaces with detailed computations of indecomposable maximal Cohen–Macaulay modules and right preresolutions is included as well.

Introduction

Let R be a commutative normal Gorenstein domain. Van den Bergh [2004] introduced the notion of noncommutative crepant resolutions of R. Roughly speaking, a noncommutative crepant resolution of R is an R-algebra of the form $\Lambda = \mathrm{End}_R(M)$, where M is a reflexive R-module satisfying certain homological conditions. Iyama and Reiten [2008] extended the notion of noncommutative crepant resolutions to module-finite commutative algebras over a noetherian commutative Cohen–Macaulay ring. Let R be a commutative Cohen–Macaulay equi-codimensional normal Gorenstein domain with a canonical module. It has been proven that noncommutative crepant resolutions of R, if they exist, are always derived equivalent provided $\dim R \leq 3$ (see [Iyama and Reiten 2008, Corollary 8.8; Iyama and Wemyss 2013, Theorem 1.5]).

In order to study noncommutative singularities, Qin, Wang and Zhang extended the notion of noncommutative resolutions to noncommutative algebras which are possibly not module-finite over their centers (see [Qin et al. 2019b]). Let A be a (both left and right) noetherian algebra over a field $\Bbbk$, and let ∂ be a symmetric

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dimension function of the category of right A-modules. Two right A-modules M and N are said to be s-isomorphic (see [Qin et al. 2019b]) if there is a right A-module P and two homomorphisms $f : P \to M$ and $g : P \to N$ such that the ∂-dimensions of the kernels and the cokernels of f and g are no larger than s. The following definition was given in [Qin et al. 2019b].

Definition 0.1. Let A be a (both left and right) noetherian algebra with ∂-dimension d. If there is a noetherian Auslander regular ∂-Cohen–Macaulay algebra B with ∂-dimension d and two finitely generated bimodules ${}_BM_A$ and ${}_AN_B$ such that $M \otimes_A N$ is $(d-2)$-isomorphic to B and $N \otimes_B M$ is $(d-2)$-isomorphic to A as bimodules, then B is called a *noncommutative quasiresolution* of A.

Qin, Wang and Zhang proved that noncommutative quasiresolutions of a noetherian algebra A with ∂-dimension 2 are Morita equivalent. If ∂-dimension of A is 3, then noncommutative quasiresolutions of A are derived equivalent (with further assumptions on A; see [Qin et al. 2019b, Theorem 0.6]). Thus, they generalized the corresponding results in [Iyama and Reiten 2008] and [Iyama and Wemyss 2013].

If, further, A is Auslander–Gorenstein and ∂-Cohen–Macaulay, then the algebra B in Definition 0.1 is isomorphic to the endomorphism algebra $\mathrm{End}_A(U)$ for some bimodule ${}_BU_A$ which is reflexive on both sides (see [Qin et al. 2019b, Corollary 3.13]).

Unlike the commutative case, given a noncommutative noetherian algebra A and a finitely generated right A-module U, it is usually a tough task to check whether $\mathrm{End}_A(U)$ is a noetherian algebra. In this sense, to find a noncommutative quasiresolution of a noetherian algebra is not an easy job in general.

In this paper, we only consider the noncommutative resolutions of noncommutative graded isolated singularities (see Section 2), which allows us to drop some restrictions on the algebras as given in Definition 0.1.

Now let A be a bounded below graded algebra, that is, $A = \bigoplus_{i\in\mathbb{Z}} A_i$ with $A_i = 0$ for $i \ll 0$. Assume A is right noetherian and locally finite. Let $\mathrm{gr}\, A$ be the category of finitely generated graded right A-modules, and $\mathrm{tors}\, A$ the subcategory of $\mathrm{gr}\, A$ consisting of finite-dimensional modules. Let $\mathrm{qgr}\, A = \mathrm{gr}\, A/\mathrm{tors}\, A$. We introduce the following version of noncommutative resolutions of right noetherian algebras, which is much closer to Van den Bergh's [2004] original definition of a noncommutative crepant resolution.

Definition 0.2 (more precisely, see Definition 2.1). Let A be a right noetherian graded algebra which is bounded below and locally finite with injective dimension $\mathrm{injdim}\, A_A = d < \infty$. If there is a small maximal Cohen–Macaulay (see Section 2) module M_A with $B = \mathrm{End}_A(M)$ such that

(i) $\mathrm{r.gldim}(B) = d$, where r. gldim is the right global dimension,

(ii) the functor $\underline{\mathrm{Hom}}_A(M, -) : \mathrm{gr}\, A \to \mathrm{gr}\, B$ induces an equivalence $\mathrm{qgr}\, A \cong \mathrm{qgr}\, B$,

then we call B a *right preresolution* of A.

If, further, B is *right generalized Artin–Schelter regular* (see Definition 1.3), then we call B a *right quasiresolution* of A.

We have the following result (see Theorem 2.3) parallel to the ones in [Iyama and Reiten 2008; Iyama and Wemyss 2013; Qin et al. 2019b]. We remark that our proof is quite different from theirs. In fact, the algebras considered in those works are assumed to be noetherian on both sides, while we only assume the right noetherianity here.

Theorem 0.3. *Let A be a right noetherian graded algebra which is bounded below and locally finite with injective dimension* $\operatorname{injdim} A_A = 2$. *If A has a right quasiresolution*, *then*

(i) *A is CM-finite*, *that is*, *there are only finitely many nonisomorphic indecomposable maximal Cohen–Macaulay* (*MCM*) *right A-modules* (*up to degree shifts*);

(ii) *any two right quasiresolutions of A are graded Morita equivalent.*

If A is an AS-Gorenstein algebra (see Definition 1.1) which is a noncommutative isolated singularity, then the CM-finiteness will induce the existence of right preresolutions, as shown in the following results (see Theorems 3.2 and 3.6).

Theorem 0.4. *Let A be an AS-Gorenstein algebra which is a noncommutative isolated singularity.*

(i) *Let M_A be an MCM module. Then $\underline{\operatorname{End}}_A(M)$ is a right noetherian graded algebra.*

(ii) *Assume that A is CM-finite and* $\operatorname{injdim} A \geq 2$. *Let $\{P_0 = A, P_1, \ldots, P_n\}$ be the set of all the nonisomorphic indecomposable MCM modules* (*up to degree shifts*). *Let $M = \bigoplus_{i=1}^{n} P_n \oplus A$. Then $B := \underline{\operatorname{End}}_A(M)$ is a right preresolution of A.*

Let S be a quantum polynomial algebra (see Section 4), that is, S is a Koszul AS-regular algebra with Hilbert series

$$H_S(t) = \frac{1}{(1-t)^n}$$

for some $n \geq 1$. Pick a central regular element $\varpi \in A$ of degree 2. The quotient algebra $A = S/S\varpi$ is called a noncommutative quadric hypersurface. Note that a noncommutative quadric hypersurface is CM-finite if and only if it is a noncommutative isolated singularity (see [Mori and Ueyama 2019, Theorem 4.13]). Hence a noncommutative quadric hypersurface A which is also a noncommutative isolated singularity always admits a right preresolution, and to find such a preresolution it suffices to compute all the indecomposable MCM-modules of A.

Assume A is a noncommutative quadric hypersurface with injective dimension d. Let $\Omega^d(\Bbbk_A)$ be the d-th syzygy of the trivial module $\Bbbk_A$. Set

$$\mathbb{M} := \Omega^d(\Bbbk_A)(d).$$

Then $\mathbb{M}$ is a Koszul MCM module. Associated to A, Smith and Van den Bergh constructed a finite-dimensional algebra $C(A)$, and proved that the stable category of MCM modules over A is equivalent to the derived category of $C(A)$. In this paper, we prove the following results, which provides a relatively easy way to compute the algebra $C(A)$ and to find all the indecomposable MCM modules of A. Consequently, we show a method to construct a right preresolution of A in case A is a CM-finite.

Theorem 0.5 (see Theorems 4.6 and 4.11). *Let S be a quantum polynomial algebra, and let $\varpi \in A$ be a central regular element of degree* 2. *Set $A := S/S\varpi$. Then*:

(i) $\mathrm{End}_{\mathrm{gr}\,A}(\mathbb{M}) \cong C(A)$.

(ii) *A is a noncommutative isolated singularity if and only if* $\mathrm{End}_{\mathrm{gr}\,A}(\mathbb{M})$ *is semisimple.*

(iii) *Assume A is a noncommutative isolated singularity. Then $B = \underline{\mathrm{End}}_A(\mathbb{M} \oplus A)$ is a right preresolution of A.*

We remark that the second statement of the above theorem follows from [He and Ye 2019, Theorem 6.3] (see also [Mori and Ueyama 2019, Theorem 4.13]).

In view of Theorem 0.5, some properties of indecomposable MCM modules of A are obtained in Section 4. We provide a concrete example of quadric hypersurfaces with detailed computations of indecomposable MCM modules and right preresolutions in the last section.

1. Preliminaries

Throughout, $\Bbbk$ will be a field of characteristic zero and all algebras considered are over $\Bbbk$.

Let A be a $\mathbb{Z}$-graded $\Bbbk$-algebra. We denote by $\mathrm{Gr}\,A$ the category of graded right A-modules, and by $\mathrm{Hom}_{\mathrm{Gr}\,A}(M, N)$ the set of homogeneous right A-module homomorphisms which preserve the degrees of elements for $M, N \in \mathrm{Gr}\,A$. For $k \in \mathbb{Z}$, $M(k)$ is the graded right A-module such that $M(k)_n = M_{n+k}$. We write $\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\,A}(M, N(k))$, and $\underline{\mathrm{Ext}}^i_A(M, N)$ for the derived functor of $\underline{\mathrm{Hom}}_A(M, N)$. In particular, we write $\underline{\mathrm{End}}_A(M)$ for $\underline{\mathrm{Hom}}_A(M, M)$.

Let $\mathrm{PHom}_{\mathrm{Gr}\,A}(M, N)$ be the subset of $\mathrm{Hom}_{\mathrm{Gr}\,A}(M, N)$ consisting of homomorphisms which factor through some graded projective modules, and let

$$\underline{\mathrm{PHom}}_A(M, N) = \bigoplus_{k\in\mathbb{Z}} \mathrm{PHom}_{\mathrm{Gr}\,A}(M, N(k)).$$

The set of graded stable homomorphism is denoted by

$$\underline{\mathrm{SHom}}_A(M, N) = \underline{\mathrm{Hom}}_A(M, N)/\,\underline{\mathrm{PHom}}_A(M, N).$$

Let A be a right noetherian graded algebra, and let $\mathrm{gr}\, A$ be the full subcategory of $\mathrm{Gr}\, A$ consisting of finitely generated modules. A graded right module $M \in \mathrm{Gr}\, A$ is called a *torsion* module if for every $m \in M$, the right submodule mA is finite-dimensional. Let $\mathrm{Tors}\, A$ be the full subcategory of $\mathrm{Gr}\, A$ consisting of all the torsion modules, and let $\mathrm{tors}\, A$ be the full subcategory of $\mathrm{Tors}\, A$ consisting of finite-dimensional objects. Since A is right noetherian, $\mathrm{Tors}\, A$ (resp. $\mathrm{tors}\, A$) is a Serre subcategory of $\mathrm{Gr}\, A$ (resp. $\mathrm{gr}\, A$). The quotient categories $\mathrm{QGr}\, A = \mathrm{Gr}\, A/\,\mathrm{Tors}\, A$ and $\mathrm{qgr}\, A = \mathrm{gr}\, A/\,\mathrm{tors}\, A$ are both abelian and $\mathrm{qgr}\, A$ is an abelian subcategory of $\mathrm{QGr}\, A$. Let $\pi : \mathrm{Gr}\, A \to \mathrm{QGr}\, A$ be the projection functor. Then π has a right adjoint functor $\omega : \mathrm{QGr}\, A \to \mathrm{Gr}\, A$ such that $\pi\omega \cong \mathrm{id}$.

For $M \in \mathrm{Gr}\, A$, we write $\mathcal{M} = \pi(M)$. The Hom-sets in $\mathrm{QGr}\, A$ is defined by

$$\mathrm{Hom}_{\mathrm{QGr}\, A}(\mathcal{M}, \mathcal{N}) = \varinjlim \mathrm{Hom}_{\mathrm{Gr}\, A}(M', N/\,\Gamma(N)),$$

where $\Gamma(N)$ is the maximal torsion submodule of N, and the limit runs over all the submodules $M' \subseteq M$ such that M/M' is a torsion module. We refer to [Artin and Zhang 1994] for more information about the quotient categories $\mathrm{QGr}\, A$ and $\mathrm{qgr}\, A$.

A graded algebra A is locally finite if $\dim A_n < \infty$ for all $n \in \mathbb{Z}$, and A is *bounded below* if $A_n = 0$ for all $n \ll 0$. If $A_0 = \Bbbk$ and $A_n = 0$ for all $n < 0$, then A is said to be *connected* graded.

We recall the following classical definition (see [Artin and Schelter 1987]).

Definition 1.1. Let A be a (both left and right) noetherian connected graded algebra. A is called an *Artin–Schelter Gorenstein* algebra of dimension d if

(i) $\mathrm{injdim}_A A = \mathrm{injdim}\, A_A = d < \infty$;

(ii) $\underline{\mathrm{Ext}}^i_A(\Bbbk_A, A_A) = 0$ for all $i \neq d$, and $\underline{\mathrm{Ext}}^d_A(\Bbbk_A, A_A) \cong {}_A\Bbbk(l)$ for some l;

(iii) the left version of (ii) is satisfied.

If, further, $\mathrm{gldim}\, A = d$, then A is called an *Artin–Schelter regular* algebra. The integer l is usually called the *Gorenstein parameter* of A.

Note that we do not require the finiteness assumption on the Gelfand–Kirillov dimension of A in the above definition. We remark that in the original definition (see [Artin and Schelter 1987]), A is assumed to have finite Gelfand–Kirillov dimension.

We need a more general version of Artin–Schelter Gorenstein algebras in this paper. If not otherwise stated, we always assume that A is a right noetherian graded algebra which is locally finite and bounded below. Let $J(A)$ be the graded Jacobson radical of A. The following properties are well-known to experts; see Lemma 3.2 and Proposition 3.3 of [Chan et al. 2019] for instance.

Lemma 1.2. *Retain the notation as above.*

(i) $A/J(A)$ *is finite-dimensional.*

(ii) $J(A_0) = J(A) \cap A_0$, *where* $J(A_0)$ *is the Jacobson radical of* A_0.

(iii) $J(A) \supseteq A_{\geq n_0}$ *for some integer* n_0, *and* $\bigcap_{n\geq 0} J(A)^n = 0$.

We generalize Artin–Schelter Gorenstein algebras to right noetherian bounded below algebras.

Definition 1.3. Let A be as above and $J = J(A)$ be the graded Jacobson radical of A. We call A a *right generalized Artin–Schelter Gorenstein* algebra of dimension d if

(i) $\operatorname{injdim} A_A = d < \infty$,

(ii) $\underline{\operatorname{Ext}}^i_A(A/J, A) = 0$ for all $i \neq d$,

(iii) $\underline{\operatorname{Ext}}^d_A(A/J, A)$ is annihilated by J when viewed as a left A-module, and $\underline{\operatorname{Ext}}^d_A(A/J, A)$ is invertible as a graded A/J-A/J-bimodule.

If, further, $\operatorname{r.gldim}(A) = d$, then we call A a *right generalized Artin–Schelter regular* algebra.

We abbreviate “Artin–Schelter” to AS, and “generalized Artin–Schelter” to GAS.

Remark 1.4. The notion of GAS-Gorenstein algebras is a slight generalization of the ones in [Reyes and Rogalski 2021, Definition 1.4] and [Minamoto and Mori 2011, Definition 3.1], where the algebras considered are $\mathbb{N}$-graded. In [Chan et al. 2019, Definition 3.9], the authors assume that the algebra A is noetherian, and admits a balanced dualizing complex.

Let $\Gamma : \operatorname{gr} A \to \operatorname{gr} A$ be the torsion functor, that is,

$$\Gamma(M) = \{m \in M \mid \dim_{\Bbbk}(mA) < \infty\}.$$

The i-th right derived functor of Γ is denoted by $R^i\Gamma$. By Lemma 1.2, we have $\Gamma \cong \lim_{n\to\infty} \underline{\operatorname{Hom}}_A(A/J^n, -)$. For $M \in \operatorname{gr} A$, the *depth* of M is defined to be

$$\operatorname{depth}(M) = \min\{i \mid R^i\Gamma(M) \neq 0\}.$$

Then $\operatorname{depth}(M)$ is either a nonnegative integer or ∞. The following lemma is classical for connected graded algebras, and the proof for connected case applies also to our case.

Lemma 1.5. $$\operatorname{depth}(M_A) = \min\{i \mid \underline{\operatorname{Ext}}^i_A(A/J, M) \neq 0\}.$$

Recall that $\pi : \operatorname{Gr} A \to \operatorname{QGr} A$ has a right adjoint functor $\omega : \operatorname{QGr} A \to \operatorname{Gr} A$.

Lemma 1.6. *Let* M_A *be a finitely generated module. If* $\operatorname{depth}(M_A) \geq 2$, *then* $\omega\pi(M) \cong M$.

Proof. Rewriting the exact sequence (3.12.3) of [Artin and Zhang 1994], we obtain the following exact sequence

$$0 \to \Gamma(M) \to M \to \omega\pi(M) \to R^1\Gamma(M) \to 0.$$

Since $\operatorname{depth}(M_A) \geq 2$, it follows that $\omega\pi(M) \cong M$. □

A useful homological identity in the theory of AS-Gorenstein algebras is the Auslander–Buchsbaum formula (see [Jørgensen 1998, Theorem 3.2] for a noncommutative version), which provides an effective way to calculate the depth of a module over a local ring. For our purpose, it will be helpful to have a more general version of the Auslander–Buchsbaum formula for right GAS-Gorenstein algebras. We mention that the proof is a modification of that of [Jørgensen 1998, Theorem 3.2].

Theorem 1.7 (Auslander–Buchsbaum formula). *Let A be a right GAS-Gorenstein algebra, and $M_A \in \operatorname{gr} A$. Suppose* $\operatorname{projdim}(M_A) < \infty$. *Then*

$$\operatorname{projdim}(M_A) + \operatorname{depth}(M_A) = \operatorname{depth}(A_A).$$

Proof. If $\operatorname{injdim} A_A = 0$, then it is clear. Assume $\operatorname{injdim}(A_A) = d \geq 1$ and $\operatorname{projdim}(M_A) = p$. Take a graded projective resolution of the right module A/J:

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to A/J \to 0,$$

where each P^i is finitely generated for all i. Applying $\underline{\operatorname{Hom}}_A(-, A)$ to the resolution, we obtain the sequence

$$0 \to \underline{\operatorname{Hom}}_A(P^0, A) \to \underline{\operatorname{Hom}}_A(P^{-1}, A) \to \underline{\operatorname{Hom}}_A(P^{-2}, A) \to \cdots . \tag{1.7.1}$$

Take a minimal graded projective resolution

$$0 \to Q^{-p} \to \cdots \to Q^{-1} \to Q^0 \to M \to 0 \tag{1.7.2}$$

of M. By taking the tensor product of (1.7.2) and (1.7.1) we obtain a double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Q^{-2} \otimes_A \underline{\operatorname{Hom}}_A(P^0, A) & \longrightarrow & Q^{-2} \otimes_A \underline{\operatorname{Hom}}_A(P^{-1}, A) & \longrightarrow & Q^{-2} \otimes_A \underline{\operatorname{Hom}}_A(P^{-2}, A) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
Q^{-1} \otimes_A \underline{\operatorname{Hom}}_A(P^0, A) & \longrightarrow & Q^{-1} \otimes_A \underline{\operatorname{Hom}}_A(P^{-1}, A) & \longrightarrow & Q^{-1} \otimes_A \underline{\operatorname{Hom}}_A(P^{-2}, A) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
Q^{0} \otimes_A \underline{\operatorname{Hom}}_A(P^0, A) & \longrightarrow & Q^{0} \otimes \underline{\operatorname{Hom}}_A(P^{-1}, A) & \longrightarrow & Q^{0} \otimes \underline{\operatorname{Hom}}_A(P^{-2}, A) & \longrightarrow & \cdots
\end{array}$$

Temporarily write X for the total complex of the above double complex. Denote by $F^I X$ the filtration on X defined by the rows, and by $F^{II} X$ the filtration on X

defined by the columns. The second page of the spectral sequence induced by the filtration $F^I X$ is

$$E_2^{rs} = \mathrm{Tor}_r^A(M, \underline{\mathrm{Ext}}_A^s(A/J, A)).$$

Note that P^i and Q^j are finitely generated projective modules for all i and j. We have natural isomorphisms

$$Q^i \otimes_A \underline{\mathrm{Hom}}_A(P^j, A) \cong \underline{\mathrm{Hom}}_A(P^j, Q^i) \quad \text{for all } i \text{ and } j.$$

Hence each column of the above diagram is exact except at the final position, whose homology group is isomorphic to $\underline{\mathrm{Hom}}_A(P^j, M)$. Therefore, the spectral sequence induced by the filtration $F^{II} X$ collapses at the first page, and the second page of this spectral sequence is

$$\underline{\mathrm{Ext}}_A^j(A/J, M), \quad j \geq 0.$$

Hence we obtain a convergent spectral sequence

(1.7.3) $$E_2^{rs} = \mathrm{Tor}_r^A(M, \underline{\mathrm{Ext}}_A^s(A/J, A)) \Rightarrow \underline{\mathrm{Ext}}_A^{s-r}(A/J, M).$$

By assumption, $\underline{\mathrm{Ext}}_A^s(A/J, A) = 0$ for $s \neq d$ and $\underline{\mathrm{Ext}}_A^d(A/J, A) \cong V$ for some graded invertible A/J-A/J-bimodule V. Thus the spectral sequence (1.7.3) collapses at the second page. Since the resolution (1.7.2) is minimal, we have $\mathrm{Tor}_r^A(M, V) \cong Q^{-r} \otimes_A V$, and hence $\mathrm{Tor}_r^A(M, V) \neq 0$ for each $0 \leq r \leq p$ for V is invertible. Therefore, $\underline{\mathrm{Ext}}_A^{d-p}(A/J, M) \neq 0$ and $\underline{\mathrm{Ext}}_A^i(A/J, M) = 0$ for all $i < d-p$. By Lemma 1.5, $\mathrm{depth}(M) = d-p$. The Auslander–Buchsbaum formula follows. □

2. Noncommutative resolutions

Noncommutative crepant resolutions for commutative Gorenstein algebras were introduced in [Van den Bergh 2004]. Noncommutative quasiresolutions for noncommutative Auslander–Gorenstein algebras were introduced in [Qin et al. 2019b]. In this section, we will modify the definition of noncommutative resolutions of [Qin et al. 2019b] and give an alternative version of noncommutative resolutions for noncommutative isolated singularities.

Let A be a right noetherian graded algebra which is bounded below and locally finite. Recall that A is called a *noncommutative isolated singularity* (see [Ueyama 2013]), if qgr A has finite global dimension, i.e., there is an integer $n_0 \geq 0$ such that $\mathrm{Ext}^i_{\mathrm{qgr}\,A}(\pi(M), \pi(N)) = 0$ for all $i > n_0$ and $M, N \in \mathrm{gr}\,A$.

A finitely generated graded right A-module M is said to be *small* if the following conditions are satisfied:

(i) $\underline{\mathrm{End}}_A(M)$ is a right noetherian graded algebra, and

(ii) $\underline{\mathrm{Hom}}_A(M, N)$ is a finitely generated graded right $\underline{\mathrm{End}}_A(M)$-module for any $N \in \mathrm{gr}\,A$.

Let M_A be a small A-module and let $B = \underline{\mathrm{End}}_A(M)$. Since M is finitely generated and A is bounded below and locally finite, B is also bounded below and locally finite, and we have an additive functor

$$F = \underline{\mathrm{Hom}}_A(M, -) : \mathrm{gr}\, A \to \mathrm{gr}\, B.$$

If K is a finite-dimensional graded right A-module, then $\underline{\mathrm{Hom}}_A(M, K)$ is a finite-dimensional graded right B-module. For $X, Y \in \mathrm{gr}\, A$, let $f : X \to Y$ be a homomorphism such that both $\ker f$ and $\operatorname{coker} f$ are finite-dimensional. Let $f_* = \underline{\mathrm{Hom}}_A(M, f)$. It is not hard to see both $\ker f_*$ and $\operatorname{coker} f_*$ are finite-dimensional. Therefore, the functor F induces a functor $\mathcal{F} : \mathrm{qgr}\, A \to \mathrm{qgr}\, B$ which fits into the commutative diagram

(2.0.1)
$$\begin{tikzcd} \mathrm{gr}\, A \arrow[r, "F"] \arrow[d, "\pi"'] & \mathrm{gr}\, B \arrow[d, "\pi"] \\ \mathrm{qgr}\, A \arrow[r, "\mathcal{F}"] & \mathrm{qgr}\, B \end{tikzcd}$$

Suppose that A has finite injective dimension $\operatorname{injdim} A_A = d$. Recall that a finitely generated graded right A-module M is called a *maximal Cohen–Macaulay module* (MCM module, for simplicity) if $R^i\Gamma(M) = 0$ for all $i \neq d$.

Definition 2.1. Let A be a right noetherian graded algebra which is bounded below and locally finite with injective dimension $\operatorname{injdim} A_A = d < \infty$. If there is a small MCM module M_A such that

(i) $\mathrm{r.gldim}(B) = d$, where $B = \underline{\mathrm{End}}_A(M)$,

(ii) the functor $\mathcal{F}$, as in the diagram (2.0.1), is an equivalence,

then we call B a *right preresolution* of A.

If, further, B is right GAS-regular, then we call B a *right quasiresolution* of A.

Remark 2.2. (1) A right noetherian graded algebra which admits a right preresolution is automatically a noncommutative isolated singularity. This follows from the well-known fact that the global dimension of $\mathrm{qgr}\, B$ is not greater than that of $\mathrm{gr}\, B$, see for instance [Artin and Zhang 1994, Section 7].

(2) The above definition is a modification of [Qin et al. 2019b, Definition 0.5] for noncommutative isolated singularities, where the algebra B is assumed to be a (both left and right) noetherian Auslander regular and Cohen–Macaulay $\mathbb{N}$-graded algebra. We will show some examples of right preresolutions and right quasiresolutions for right GAS-Gorenstein algebras in the subsequent sections. We consider nonpositively graded algebras because a noncommutative resolution of a noetherian algebra may not be positively graded in general.

(3) Assume that a noetherian graded locally finite algebra B is a quasiresolution of a noetherian graded algebra A in sense of Qin, Wang and Zhang. If the dimension function ∂ in [Qin et al. 2019b, Definition 0.5] is taken to be the Gelfand–Kirillov dimension, then, by [loc. cit., Lemma 1.9 and Corollary 3.13], B is a right preresolution of A in our sense. Besides, the Auslander regularity and Cohen–Macaulay hypothesis on a quasiresolution B in [loc. cit., Definition 0.5] imply that B is a GAS-regular algebra (see [Qin et al. 2019a, Lemma 1.10]). Therefore, if the dimension function ∂ in [Qin et al. 2019b, Definition 0.5] is taken to be the Gelfand–Kirillov dimension, then the concept of noncommutative quasiresolutions defined in [Qin et al. 2019b] implies right quasiresolution in our sense.

(4) Quasiresolutions of invariant subalgebras of finite group actions on regular algebras usually satisfy the conditions in (3). For example, let $S = \Bbbk_{-1}[x_1, \dots, x_n]$ with $n \geq 2$, the skew polynomial algebra. Assume that G is a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(S)$. If G is *small* (i.e., G does not contain a psuedoreflection when viewed as a subgroup of $\mathrm{GL}(V)$, where $V = \mathrm{span}\{x_1, \dots, x_n\}$), then by [Bao et al. 2019, Theorem 2.6] and [Bao et al. 2018, Theorem 0.8], the skew group algebra $B(:= S * G)$ is a quasiresolution both in sense of Qin, Wang and Zhang and of ours.

A right noetherian graded locally finite algebra with finite right injective dimension is said to be *CM-finite* if it has, up to degree shifts, finitely many nonisomorphic indecomposable MCM modules.

Our main result of this section is as follows. It may be viewed as a noncommutative version of [Iyama and Wemyss 2013, Theorem 1.5] in the dimension 2 case.

Theorem 2.3. *Let A be a right noetherian graded algebra which is bounded below and locally finite with injective dimension* $\operatorname{injdim} A_A = 2$. *Assume that A has a right quasiresolution. Then*

(i) *A is CM-finite*;

(ii) *any two right quasiresolutions of A are graded Morita equivalent.*

Proof. (i) By assumption, there exists some small MCM module M_A such that $B = \underline{\mathrm{End}}_A(M)$ is a right quasiresolution of A. Consider the functor

$$F = \underline{\mathrm{Hom}}_A(M, -) : \mathrm{gr}\, A \to \mathrm{gr}\, B$$

and the induced functor $\mathcal{F} : \mathrm{qgr}\, A \to \mathrm{qgr}\, B$. Then we have commutative diagram (2.0.1). Take an MCM A-module N, and let $X = \underline{\mathrm{Hom}}_A(M, N)$. Then $X \in \mathrm{gr}\, B$. Since $\mathcal{F}$ is an equivalence, we have

$$\mathrm{Hom}_{\mathrm{qgr}\, A}(\pi(M), \pi(N)(k)) \cong \mathrm{Hom}_{\mathrm{qgr}\, B}(\mathcal{F}(\pi(M)), \mathcal{F}(\pi(N)(k))), \text{ for all } k \in \mathbb{Z}.$$

By the commutativity of (2.0.1),

$$
\begin{aligned}
\operatorname{Hom}_{\operatorname{qgr} B}(\mathcal{F}(\pi(M)), \mathcal{F}(\pi(N)(k))) \\
&= \operatorname{Hom}_{\operatorname{qgr} B}(\pi(\underline{\operatorname{Hom}}_A(M, M)), \pi(\underline{\operatorname{Hom}}_A(M, N))(k)) \\
&= \operatorname{Hom}_{\operatorname{qgr} B}(\pi(B), \pi(X)(k)).
\end{aligned}
$$

Let J be the graded Jacobson radical of B. Since B/J is finite-dimensional and X is finitely generated, we have

$$\operatorname{Hom}_{\operatorname{qgr} B}(\pi(B), \pi(X)(k)) = \lim_{n\to\infty} \operatorname{Hom}_{\operatorname{gr} B}(J^n, X(k)).$$

From the exact sequence $0 \to J^n \to B \to B/J^n \to 0$ we obtain the exact sequence

$$
\begin{aligned}
(2.3.1)\quad 0 \to \lim_{n\to\infty} \operatorname{Hom}_{\operatorname{gr} B}(B/J^n, X(k)) \to \operatorname{Hom}_{\operatorname{gr} B}(B, X(k)) \\
\to \lim_{n\to\infty} \operatorname{Hom}_{\operatorname{gr} B}(J^n, X(k)) \to \lim_{n\to\infty} \operatorname{Ext}^1_{\operatorname{gr} B}(B/J^n, X(k)) \to 0.
\end{aligned}
$$

By the commutative diagram (2.0.1), we have the following commutative diagram:

$$
\begin{tikzcd}
\operatorname{Hom}_{\operatorname{gr} B}(B, X(k)) \arrow[r] & \lim_{n\to\infty} \operatorname{Hom}_{\operatorname{gr} B}(J^n, X(k)) \\
\operatorname{Hom}_{\operatorname{gr} B}(B, X(k)) \arrow[r] \arrow[u, "="] & \operatorname{Hom}_{\operatorname{qgr} B}(\pi(B), \pi(X)(k)) \arrow[u, "="] \\
\operatorname{Hom}_{\operatorname{gr} A}(M, N(k)) \arrow[r] \arrow[u, "F"] & \operatorname{Hom}_{\operatorname{qgr} A}(\pi(M), \pi(N)(k)). \arrow[u, "\mathcal{F}"]
\end{tikzcd}
\tag{2.3.2}
$$

Note that $\operatorname{Hom}_{\operatorname{gr} B}(B, X(k)) = X_k = \operatorname{Hom}_{\operatorname{gr} A}(M, N(k))$. It follows that the maps in the left column of the diagram (2.3.2) are bijective. Since $\mathcal{F}$ is an equivalence, the maps in the right column are bijective. Since N is MCM, then $\operatorname{depth}(N) = 2$. It follows from Lemma 1.6 that $\operatorname{Hom}_{\operatorname{qgr} A}(\pi(M), \pi(N)(k)) \cong \operatorname{Hom}_{\operatorname{gr} A}(M, N(k))$. Therefore the bottom map in the diagram (2.3.2) is also bijective. Hence the top map in the diagram (2.3.2) is an isomorphism.

By the exact sequence (2.3.1), we have, $\lim_{n\to\infty} \operatorname{Hom}_{\operatorname{gr} B}(B/J^n, X(k)) = 0$ and $\lim_{n\to\infty} \operatorname{Ext}^1_{\operatorname{gr} B}(B/J^n, X(k)) = 0$ for all $k \in \mathbb{Z}$. Hence $\Gamma(X) = R^1\Gamma(X) = 0$. It follows that $\operatorname{depth}(X) \geq 2$. Since $\operatorname{r.gldim}(B) = \operatorname{injdim} A_A = 2$, we have that X is a projective B-module by the Auslander–Buchsbaum formula (see Theorem 1.7). Hence $X \in \operatorname{add}(B)$, where $\operatorname{add}(B)$ is the category of direct summands of direct sums of finite copies of degree shifts of B. Therefore $\pi(X) \in \operatorname{add}(\pi(B))$ and hence $\pi(N) \in \operatorname{add}(\pi(M))$ for $\mathcal{F}$ is an equivalence. Since both M and N are MCM, it follows that $\omega\pi(M) \cong M$ and $\omega\pi(N) \cong N$ (see Lemma 1.6). Hence $N \in \operatorname{add}(M)$, which implies that every indecomposable MCM module is a direct summand of M (up to a degree shift). Hence A is CM-finite.

(ii) Suppose that M' is another small MCM module such that $B' = \underline{\mathrm{End}}_A(M')$ is a right quasiresolution of A. By the proof of (i), $M' \in \mathrm{add}(M)$ and $M \in \mathrm{add}(M')$. Therefore $\mathrm{add}(M) = \mathrm{add}(M')$. Hence B and B' are graded Morita equivalent. □

Remark 2.4. In [Qin et al. 2019b], the authors extended the theory of noncommutative crepant resolutions for commutative algebras to noncommutative settings. They proved that noncommutative quasiresolutions for a noetherian $\mathbb{N}$-graded algebra with Gelfand–Kirillov dimension 2 are always Morita equivalent (see [Qin et al. 2019b, Theorem 0.6(1)]), which extensively generalizes a similar result in commutative case (see [Iyama and Wemyss 2013, Theorem 1.5]). We remark that a noncommutative quasiresolution B in [Qin et al. 2019b] is assumed to be (left and right) noetherian $\mathbb{N}$-graded Auslander regular and Cohen–Macaulay. In contrast, we assume that the resolution B is right GAS-regular. Moreover, the method we used is different from that of [Qin et al. 2019b].

3. Endomorphism rings of CM-finite AS-Gorenstein algebras

In this section, A is an AS-Gorenstein algebra. We will show that the endomorphism ring of an MCM module over a noncommutative isolated singularity is always right noetherian, which suggests the existence of resolutions for noncommutative isolated singularities.

Lemma 3.1. *Let A be an AS-Gorenstein algebra which is a noncommutative isolated singularity. Let M_A be an MCM module, and let N_A be a finitely generated graded A-module. Then $\underline{\mathrm{SHom}}_A(M, N)$ is finite-dimensional.*

Proof. Assume that the injective dimension $\mathrm{injdim}(A_A) = d$. Since M is an MCM module, there is an exact sequence

$$0 \to M \xrightarrow{\tau} P^{-d} \xrightarrow{\partial^{-d}} P^{-d+1} \to \cdots \to P^0 \xrightarrow{\partial^0} P^1 \to \cdots,$$

where P^i is a finitely generated graded projective A-module for all $i \geq -d$. Let $X = \mathrm{im}\,\partial^0$. Then

$$\underline{\mathrm{Ext}}_A^{d+1}(X, N) = \underline{\mathrm{Hom}}_A(M, N)/\,\mathrm{im}(\tau^*),$$

where $\tau^* : \underline{\mathrm{Hom}}_A(P^{-d}, N) \to \underline{\mathrm{Hom}}_A(M, N)$ is the induced map. Moreover $\mathrm{im}(\tau^*) = \underline{\mathrm{PHom}}_A(M, N)$ since M is an MCM module. Hence

$$\underline{\mathrm{Ext}}_A^{d+1}(X, N) = \underline{\mathrm{Hom}}_A(M, N)/\,\mathrm{im}(\tau^*) = \underline{\mathrm{SHom}}_A(M, N).$$

By assumption A is a noncommutative isolated singularity, and hence $\underline{\mathrm{Ext}}_A^{d+1}(X, N)$ is finite-dimensional (see [Smith and Van den Bergh 2013, Proposition 4.3]). □

We fix a notation which will be used in the proof of the next theorem. For an integer i and a graded A-module X, we define a graded homomorphism $s^i : X(i) \to X$ by setting $s^i(x) = x$ for all $x \in X$. Then s^i is a graded homomorphism of degree i.

Theorem 3.2. *Let A be an AS-Gorenstein algebra which is a noncommutative isolated singularity. Let M_A be an MCM module. Then $\underline{\mathrm{End}}_A(M)$ is a right noetherian graded algebra.*

Proof. Set $B = \underline{\mathrm{End}}_A(M)$. Let I be a graded right ideal of B and let S be a finite set consisting of homogeneous elements of I. We write $M^S = \sum_{b\in S} b(M)$. Then M^S is a submodule of M. Consider the set

$$\mathcal{X} = \{M^S \mid S \text{ is a finite set of homogeneous elements of } I\}.$$

Since A is noetherian and M is finitely generated, the set $\mathcal{X}$ has a maximal object. Let M^{S_0} be a maximal object in $\mathcal{X}$. For every element $b' \in I$, consider the set $S_1 = S_0 \cup \{b'\}$. Since $M^{S_1} \supseteq M^{S_0}$, and by assumption M^{S_0} is maximal, it follows that $M^{S_1} = M^{S_0}$. Hence $b'(M) \subseteq \sum_{b\in S_0} b(M)$ and $M^{S_0} = IM$. Set $N = IM = M^{S_0}$.

Assume that $S_0 = \{b_1, \dots, b_n\}$, and that the degrees of $b_1, \dots, b_n$ are $k_1, \dots, k_n$ respectively. Let

$$\phi : M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n) \to N$$

be the homomorphism defined by the homomorphisms $\{b_1 s^{-k_1}, \dots, b_n s^{-k_n}\}$. Then ϕ is a homomorphism of degree 0.

We next prove that the right ideal I is finitely generated. There are two different situations.

Case 1. For $b' \in I$, assume the degree of the homomorphism b' is k, and assume that the composition $M(-k) \xrightarrow{s^{-k}} M \xrightarrow{b'} N$ factors through a graded projective module, that is, there is a graded projective module P such that $b's^{-k}$ is the composition $M(-k) \xrightarrow{r} P \xrightarrow{t} N$, where both r and t are homomorphisms of degree 0. Since ϕ is an epimorphism, there is a homomorphism

$$f : P \to M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n)$$

such that $t = \phi f$. Then $b's^{-k} = tr = \phi f r$. Let $h = fr$. Then h is a morphism from $M(-k)$ to $M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n)$. Let

$$h' : M \to M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n)$$

be the homomorphism such that $h = h's^{-k}$. Then $b' = \phi h'$. Let

$$p_i : M(-k_1) \oplus M(-k_2) \oplus \cdots \oplus M(-k_n) \to M(-k_i)$$

be the projection map. Then $b' = \sum_{i=1}^n (b_i s^{-k_i})(p_i h')$. For each i, let $h_i' = s^{-k_i} p_i h'$. Then h_i' is an endomorphism of M. Hence, in this case, $b' = \sum_{i=1}^n b_i h_i' \in \sum_{i=1}^n b_i B$.

Case 2. Assume b' does not factor through any projective modules. Since A is a noncommutative isolated singularity, $\underline{\mathrm{SHom}}_A(M, N)$ is finite-dimensional by Lemma 3.1. Note that $\mathrm{im}(b) = bM \subseteq N$ for any $b \in I$, thus we may view b as an element in $\mathrm{Hom}_A(M, N)$ and identify I with a subspace of $\mathrm{Hom}_A(M, N)$. Then

$$\bar{I} = I/(I \cap \underline{\mathrm{PHom}}_A(M, N)) \cong (I + \underline{\mathrm{PHom}}_A(M, N))/\,\underline{\mathrm{PHom}}_A(M, N)$$

is finite-dimensional, for the latter is a subspace of $\underline{\mathrm{SHom}}_A(M, N)$. Choose homogeneous elements $f_1, \dots, f_m \in I$ such that their images $\bar{f}_1, \dots, \bar{f}_m$ in $\bar{I}$ form a basis of $\bar{I}$. Let $\overline{b'}$ be the image of b' in the quotient space $\bar{I}$. Since $\bar{f}_1, \dots, \bar{f}_m$ is a basis, we may write $\overline{b'} = l_1 \bar{f}_1 + \cdots + l_m \bar{f}_m$ for some $l_1, \dots, l_m \in \Bbbk$. Then $b' - (l_1 f_1 + \cdots + l_m f_m) \in I \cap \underline{\mathrm{PHom}}_A(M, N)$. By Case 1, $b' - (l_1 f_1 + \cdots + l_m f_m) = \sum_{i=1}^n b_i g_i$ for some $g_1, \dots, g_n \in B$. Hence $b' = l_1 f_1 + \cdots + l_m f_m + \sum_{i=1}^n b_i g_i$.

Summarizing, the right ideal I is generated by $f_1, \dots, f_m, b_1, \dots, b_n$. □

The proof of Theorem 3.2 also implies the following result.

Corollary 3.3. *Retain the same notation as in Theorem 3.2. Let N be a finitely generated graded right A-module. Then $\underline{\mathrm{Hom}}_A(M, N)$ is a finitely generated graded right $\underline{\mathrm{End}}_A(M)$-module.*

We conclude the following result which suggests noncommutative resolutions for noncommutative singularities.

Proposition 3.4. *Let A be an AS-Gorenstein algebra and $M_A \in \mathrm{gr}\, A$ be an MCM module. Assume that A is a noncommutative isolated singularity. Then $B := \underline{\mathrm{End}}_A(M \oplus A)$ is a right noetherian graded algebra, and moreover, there is an equivalence of abelian categories $\mathrm{qgr}\, B \cong \mathrm{qgr}\, A$.*

Proof. The graded algebra B may be written as a matrix algebra

$$B = \begin{pmatrix} \underline{\mathrm{End}}_A(M) & M \\ M^\vee & A \end{pmatrix},$$

where $M^\vee = \underline{\mathrm{Hom}}_A(M, A)$. Define a map

$$\varphi : M \otimes_A M^\vee \to \underline{\mathrm{End}}_A(M), m_1 \otimes_A f \mapsto [m_2 \mapsto m_1 f(m_2)].$$

The multiplication of the above matrix algebra reads as

$$\begin{pmatrix} g_1 & m_1 \\ f_1 & a_1 \end{pmatrix}\begin{pmatrix} g_2 & m_2 \\ f_2 & a_2 \end{pmatrix} = \begin{pmatrix} g_1 g_2 + \varphi(m_1 \otimes f_2) & g_1(m_2) + m_1 a_2 \\ f_1 g_2 + a_1 f_1 & f_1(m_2) + a_1 a_2 \end{pmatrix}$$

Let $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $A \cong eBe$, and

$$BeB = \begin{pmatrix} \varphi(M \otimes_A M^\vee) & M \\ M^\vee & A \end{pmatrix}.$$

It is easy to see $\varphi(M \otimes_A M^\vee) = \underline{\mathrm{PHom}}_A(M, M)$. Then

$$B/BeB \cong \underline{\mathrm{SHom}}_A(M, M).$$

By Lemma 3.1, B/BeB is finite-dimensional. By the graded version of the proof of [Bao et al. 2019, Lemma 2.3], we obtain the desired equivalence $\mathrm{qgr}\, B \cong \mathrm{qgr}\, A$. $\square$

Lemma 3.5. *Let A be an AS-Gorenstein algebra which is a noncommutative isolated singularity. Let M_A be an MCM module. For $X \in \mathrm{gr}\, A$, define*

$$\varphi_X : X \otimes_A \underline{\mathrm{Hom}}_A(M, A) \to \underline{\mathrm{Hom}}_A(M, X), \quad x \otimes f \mapsto [m \mapsto xf(m)].$$

Then both $\ker(\varphi_X)$ and $\mathrm{coker}(\varphi_X)$ are finite-dimensional.

Proof. Let $0 \to K \to P \to X \to 0$ be an exact sequence with P a finitely generated graded projective A-module. Then we have the commutative diagram

$$\begin{tikzcd}
 & K \otimes_A \underline{\mathrm{Hom}}_A(M, A) \arrow[r] \arrow[d, "\varphi_K"'] & P \otimes_A \underline{\mathrm{Hom}}_A(M, A) \arrow[r] \arrow[d, "\varphi_P"'] & X \otimes_A \underline{\mathrm{Hom}}_A(M, A) \arrow[r] \arrow[d, "\varphi_X"'] & 0 \\
0 \arrow[r] & \underline{\mathrm{Hom}}_A(M, K) \arrow[r] & \underline{\mathrm{Hom}}_A(M, P) \arrow[r] & \underline{\mathrm{Hom}}_A(M, X) &
\end{tikzcd}$$

with exact rows. Note that φ_P is an isomorphism. By the snake lemma, $\ker(\varphi_X) \cong \mathrm{coker}(\varphi_K)$.

Since $\mathrm{im}(\varphi_X) = \underline{\mathrm{PHom}}_A(M, X)$, it follows from Lemma 3.1 that $\mathrm{coker}(\varphi_X)$ is finite-dimensional. Similarly $\mathrm{coker}(\varphi_K)$ and hence $\ker(\varphi_X)$ are also finite-dimensional. $\square$

Theorem 3.6. *Let A be an AS-Gorenstein algebra with $\mathrm{injdim}\, A \geq 2$ which is a noncommutative isolated singularity. Assume A is CM-finite. Let $\{P_0 = A, P_1, \ldots, P_n\}$ be the set of all the nonisomorphic indecomposable MCM modules (up to degree shifts). Let $M = \bigoplus_{i=1}^n P_n \oplus A$. Then $B := \underline{\mathrm{End}}_A(M)$ is a right preresolution of A.*

Proof. By Theorem 3.2, B is right noetherian. Since M is an MCM A-module and A is a noncommutative isolated singularity, it follows from Corollary 3.3 that $\underline{\mathrm{Hom}}_A(M, K)$ is finitely generated for every finitely generated graded right A-module K. Therefore M is a small A-module.

Set $F = \underline{\mathrm{Hom}}_A(M, -) : \mathrm{gr}\, A \to \mathrm{gr}\, B$ and $\mathcal{F} : \mathrm{qgr}\, A \to \mathrm{qgr}\, B$ to be the induced functor. We next show that $\mathcal{F}$ is an equivalence. By the proof of Proposition 3.4 (see also [Bao et al. 2019, Lemma 2.3]), the functor $G = - \otimes_A M^\vee : \mathrm{gr}\, A \to \mathrm{gr}\, B$ induces an equivalence of abelian categories $\mathcal{G} : \mathrm{qgr}\, A \to \mathrm{qgr}\, B$. Now for any $X \in \mathrm{gr}\, A$,

$$\mathcal{G}(\pi(X)) = \pi(X \otimes_A M^\vee) = \pi(X \otimes_A \underline{\mathrm{Hom}}_A(M, A)).$$

Let $\varphi_X : X \otimes_A \underline{\mathrm{Hom}}_A(M, A) \to \underline{\mathrm{Hom}}_A(M, X)$ be the map as in Lemma 3.5. Then both $\ker(\varphi_X)$ and $\mathrm{coker}(\varphi_X)$ are finite-dimensional, and φ_X induces a natural isomorphism

$$\pi(X \otimes_A \underline{\mathrm{Hom}}_A(M, A)) \cong \pi(\underline{\mathrm{Hom}}_A(M, X)).$$

It follows that $\mathcal{F}$ is naturally isomorphic to $\mathcal{G}$, and hence an equivalence.

By [Chan et al. 2019, Theorem 5.4] (see also [Leuschke 2007] for the commutative case), $\mathrm{r.gldim}(B) = d$. We remark that in [Chan et al. 2019, Theorem 5.4], the algebra A is assumed to be Cohen–Macaulay which ensures that A_A is a maximal Cohen–Macaulay module. In our case, A_A is automatically maximal Cohen–Macaulay since A is an AS-Gorenstein algebra, and then all the narratives in the proof of [Chan et al. 2019, Theorem 5.4] remain true.

Now all conditions in Definition 2.1 are satisfied. Hence B is a right preresolution of A. □

4. Noncommutative quadric hypersurfaces

In this section, we focus on noncommutative resolutions of noncommutative quadric hypersurfaces. Let us recall some terminologies.

Let A be a locally finite connected graded algebra. A graded A-module M_A is called a *Koszul module* (see [Priddy 1970]) if M_A has a *linear* projective resolution; that is, a projective resolution

$$\cdots \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to M \to 0,$$

such that P^{-n} is a graded projective module generated in degree n for all $n \geq 0$. If the trivial module $\Bbbk_A$ is a Koszul module, then A is called a *Koszul algebra*. It is known that a Koszul algebra A must be quadratic, that is, A may be written as $A = T(V)/(R)$, where V is a finite-dimensional vector space, and R is contained in $V \otimes V$. The *quadratic dual* of A is defined to be the graded algebra $A^! = T(V^*)/(R^\perp)$, where $R^\perp$ is the orthogonal dual of R in the space $V^* \otimes V^*$. Note that $A^!$ is also a Koszul algebra.

For a locally finite graded algebra A, the Hilbert series of A is defined to be the formal power series:

$$H_A(t) = \sum_{i \in \mathbb{Z}} (\dim A_i) t^i.$$

A noetherian connected graded algebra S is called a *quantum polynomial algebra* if the following conditions are satisfied:

(i) S is a Koszul AS-regular algebra;

(ii) the Hilbert series of S is $H_S(t) = \dfrac{1}{(1-t)^d}$ for some $d \geq 1$.

Let S be a quantum polynomial algebra. Suppose $w \in S_2$ is a central regular element in S of degree two. The quotient algebra $A = S/Sw$ is usually called a *noncommutative quadric hypersurface*.

The following properties of noncommutative quadric hypersufaces are well known (see [Smith and Van den Bergh 2013, Lemma 5.1(1); He and Ye 2019, Lemma 1.2] for instance). Note that for a quantum polynomial algebra the Gorenstein parameter coincides with the global dimension.

Lemma 4.1. *Assume S is a quantum polynomial algebra with global dimension $d+1$ ($d \geq 0$). Let $w \in S_2$ be a central regular element of S, and let $A = S/Sw$.*

(i) *A is a Koszul algebra.*

(ii) *A is AS-Gorenstein of injective dimension d with Gorenstein parameter $d-1$.*

(iii) *There is a central regular element $\varpi \in A^!_2$ such that $S^! \cong A^!/A^!\varpi$.*

Setup 4.2. In the rest of this section, S is a quantum polynomial algebra of global dimension $d+1$ with $d \geq 0$, and $w \in S_2$ is a central regular element. Set $A = S/Sw$.

We recall some results obtained in [Smith and Van den Bergh 2013]. Let $D^b(\operatorname{gr} A)$ be the bounded derived category of $\operatorname{gr} A$. There is a Koszul duality (see [Smith and Van den Bergh 2013, Subsection 2.4], or [Beilinson et al. 1996, Section 3] for the general situation):

$$K : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^!). \tag{4.2.1}$$

Notice that

$$K(\Bbbk) = A^!,\ K(A) = \Bbbk,\ K(M(1)) = K(M)[-1](1) \quad \text{for all } M \in D^b(\operatorname{gr} A). \tag{4.2.2}$$

The above duality induces the following duality:

$$\overline{K} : D^b(\operatorname{gr} A)/\operatorname{per} A \to D^b(\operatorname{gr} A^!)/D^b_{\text{tors}}(\operatorname{gr} A^!), \tag{4.2.3}$$

where $\operatorname{per} A$ is the full subcategory of $D^b(\operatorname{gr} A)$ consisting of perfect complexes, and $D^b_{\text{tors}}(\operatorname{gr} A^!)$ is the full subcategory of $D^b(\operatorname{gr} A^!)$ consisting of complexes with finite-dimensional total cohomology. Notice that $D^b(\operatorname{gr} A^!)/D^b_{\text{tors}}(\operatorname{gr} A^!) \cong D^b(\operatorname{qgr} A^!)$. Let $\operatorname{mcm} A$ be the full subcategory of $\operatorname{gr} A$ consisting of all the MCM modules, and let $\underline{\operatorname{mcm}}\, A$ be the stable category. Then there is a natural equivalence of triangulated categories [Buchweitz 1986, Theorem 4.4.1(2)]:

$$G : \underline{\operatorname{mcm}}\, A \to D^b(\operatorname{gr} A)/\operatorname{per} A. \tag{4.2.4}$$

Combining the functors in (4.2.3) and (4.2.4), we obtain the following, which is called the Buchweitz duality (see [Smith and Van den Bergh 2013, Theorem 3.2]):

$$B : \underline{\operatorname{mcm}}\, A \to D^b(\operatorname{qgr} A^!). \tag{4.2.5}$$

By Lemma 4.1(iii), there is a central regular element $\varpi \in A^!_2$. Let $A^![\varpi^{-1}]$ be the localization of $A^!$ by the multiplication set defined by ϖ. Then $A^![\varpi^{-1}]$ is a $\mathbb{Z}$-graded algebra. Define a finite-dimensional algebra (see [Smith and Van den Bergh 2013, Subsection 5.1]):

$$C(A) = A^![\varpi^{-1}]_0, \tag{4.2.6}$$

which is the degree zero part of $A^![\varpi^{-1}]$. Since S is a quantum polynomial algebra, it follows that (see [Smith and Van den Bergh 2013])

$$\dim C(A) = \sum_{i \geq 0} \dim S^!_{2i} = \frac{1}{2} \dim S^!. \tag{4.2.7}$$

Notice that $A^!/A^!\varpi$ is finite-dimensional since it is isomorphic to $S^!$. It follows that $M \in \operatorname{tors} A^!$ if and only if every $m \in M$ is annihilated by some power of ϖ. Therefore, there is an equivalence

$$\operatorname{qgr} A^! \to \operatorname{mod} C(A), \quad \pi(N) \mapsto N[\omega^{-1}],$$

where $\operatorname{mod} C(A)$ is the category of finite dimension modules. Therefore we have the following equivalence of triangulated categories:

$$L : D^b(\operatorname{qgr} A^!) \to D^b(\operatorname{mod} C(A)).$$

Notice that

$$L(\pi(A^!)) = C(A). \tag{4.2.8}$$

Combining L with the Buchweitz duality, we obtain the following equivalence (see [Smith and Van den Bergh 2013, Proposition 5.2]):

$$F : \underline{\operatorname{mcm}}\, A \xrightarrow{B} D^b(\operatorname{qgr} A^!) \xrightarrow{L} D^b(\operatorname{mod} C(A)). \tag{4.2.9}$$

We may put the above mentioned triangle equivalences in the following commutative diagram:

$$\begin{tikzcd}
\underline{\operatorname{mcm}}\, A \arrow[r, "B"] \arrow[d, "G"] \arrow[rr, bend left, "F"] & D^b(\operatorname{qgr} A^!) \arrow[r, "L"] \arrow[d, "\cong"] & D^b(\operatorname{mod} C(A)) \\
D^b(\operatorname{gr} A)/\operatorname{per} A \arrow[r, "\overline{K}"] & D^b(\operatorname{gr} A^!)/D^b_{\operatorname{tors}}(\operatorname{gr} A^!) &
\end{tikzcd}$$

We remark that the finite-dimensional algebra $C(A)$ may be obtained from a Clifford deformation of the Frobenius algebra $S^!$ (see [He and Ye 2019]). In this section, we will give a new method to obtain the finite-dimensional algebra $C(A)$.

Take a minimal graded projective resolution of $\Bbbk_A$ as follows:

$$\cdots \to P^{-d} \xrightarrow{\partial^{-d}} P^{-d+1} \xrightarrow{\partial^{-d+1}} \cdots \to P^0 \to \Bbbk_A \to 0. \tag{4.2.10}$$

Let $\Omega^d(\Bbbk_A) = \ker \partial^{-d+1}$ be the d-th syzygy of the trivial module. Since A is a Koszul algebra, $\Omega^d(\Bbbk_A)$ is generated in degree d. We fix notation as follows:

$$\mathbb{M} := \Omega^d(\Bbbk_A)(d).$$

We have the following properties of $\mathbb{M}$.

Lemma 4.3. *Retain the notation as above.*

(i) $\mathbb{M}$ *is a Koszul A-module.*

(ii) $\mathbb{M}$ *is an MCM module.*

Proof. (i) Since A is a Koszul algebra, the trivial module $\Bbbk_A$ has a linear projective resolution, which implies $\mathbb{M} = \Omega^d(\Bbbk_A)(d)$ has a linear projective resolution.

(ii) Since A is AS-Gorenstein of injective dimension d, it follows that $R^i\Gamma(A_A) = 0$ for $i \neq d$. By the local duality theorem (see [Van den Bergh 1997, Theorem 5.1]), $\operatorname{depth}(N) \leq d$ for every finite generated module N_A. Applying $R^i\Gamma$ to the short exact sequence

$$0 \to \Omega^j(\Bbbk_A) \to P^{-j+1} \to \Omega^{j-1}(\Bbbk_A) \to 0,$$

we obtain isomorphisms

$$R^i\Gamma(\Omega^{j-1}(\Bbbk_A)) \cong R^{i+1}\Gamma(\Omega^j(\Bbbk_A)) \quad \text{for } i < d.$$

Now by using an induction on j, we obtain that $\mathbb{M}$ is MCM. □

We investigate more properties of $\mathbb{M}$. For a graded right A-module X, let $X^\vee = \underline{\operatorname{Hom}}_A(X, A)$ denote the dual module. Clearly $X^\vee$ is a graded left A-module in an obvious way.

Proposition 4.4. *Retain the notation as above. Then $\mathbb{M}^\vee(1)$ is a left Koszul A-module.*

Proof. Applying the functor $\underline{\operatorname{Hom}}_A(-, A)$ to the exact sequence (4.2.10), we obtain the sequence

$$\begin{aligned} 0 \to \underline{\operatorname{Hom}}_A(P^0, A) &\to \underline{\operatorname{Hom}}_A(P^{-1}, A) \\ &\to \cdots \to \underline{\operatorname{Hom}}_A(P^{-d+1}, A) \xrightarrow{\iota^\vee} \underline{\operatorname{Hom}}_A(\Omega(\Bbbk_A), A) \to 0. \end{aligned} \tag{4.4.1}$$

Since A is AS-Gorenstein of injective dimension d with Gorenstein parameter $d-1$, the sequence (4.4.1) is exact except at the last position, where the cohomology group is ${}_A\Bbbk(d-1)$. Note that $\mathbb{M}^\vee(d) = \underline{\mathrm{Hom}}_A(\Omega(\Bbbk_A), A)$. The sequence (4.4.1) implies an exact sequence of left A-modules

$$0 \to K \to \mathbb{M}^\vee(d) \to {}_A\Bbbk(d-1) \to 0, \tag{4.4.2}$$

where $K = \operatorname{im} \iota^\vee$. Note that K has a projective resolution

$$0 \to \underline{\mathrm{Hom}}_A(P^0, A) \to \underline{\mathrm{Hom}}_A(P^{-1}, A) \to \cdots \to \underline{\mathrm{Hom}}_A(P^{-d+1}, A) \to K \to 0.$$

Hence $K(-d+1)$ is a left Koszul A-module. Since the subcategory of modules possessing a linear resolution is closed under extensions, the short exact sequence (4.4.2) implies the Koszulity of $\mathbb{M}^\vee(1)$. □

Proposition 4.5. *Retain the notation as above. We have the following properties*:

(i) $\underline{\mathrm{Ext}}^i_A(\Bbbk_A, \mathbb{M}) = 0$ *for* $i < d$.

(ii) *The graded vector space* $\underline{\mathrm{Ext}}^d_A(\Bbbk_A, \mathbb{M})$ *is concentrated in degree* $-d$.

Proof. (i) This follows from the fact that $\mathbb{M}$ is an MCM module.

(ii) Since A is AS-Gorenstein, $R\,\underline{\mathrm{Hom}}_A(-, A) : D^b(\mathrm{gr}\, A) \to D^b(\mathrm{gr}\, A^\circ)$ is a duality (see [Yekutieli 1992]), where A° is the opposite algebra of A. Since $\mathbb{M}$ is an MCM module, it follows that $R\,\underline{\mathrm{Hom}}_A(\mathbb{M}, A) \cong \underline{\mathrm{Hom}}_A(\mathbb{M}, A) = \mathbb{M}^\vee$ in $D^b(\mathrm{gr}\, A^\circ)$. Note that $R\,\underline{\mathrm{Hom}}_A(\Bbbk_A, A) \cong {}_A\Bbbk[-d](d-1)$. Therefore we have

$$\begin{aligned}
\underline{\mathrm{Ext}}^d_A(\Bbbk_A, \mathbb{M})_i &= \mathrm{Hom}_{D^b(\mathrm{gr}\, A)}(\Bbbk_A, \mathbb{M}[d](i)) \\
&\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A^\circ)}\big(R\,\underline{\mathrm{Hom}}_A(\mathbb{M}, A)[-d], R\,\underline{\mathrm{Hom}}_A(\Bbbk_A, A)(i)\big) \\
&\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A^\circ)}\big(\mathbb{M}^\vee, {}_A\Bbbk(d+i-1)\big) \\
&\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A^\circ)}\big(\mathbb{M}^\vee(1), {}_A\Bbbk(d+i)\big) \\
&\cong \mathrm{Hom}_{\mathrm{gr}\, A^\circ}\big(\mathbb{M}^\vee(1), {}_A\Bbbk(d+i)\big).
\end{aligned}$$

By Proposition 4.4, $\mathbb{M}^\vee(1)$ is generated in degree zero. Thus for $i \neq -d$, we have $\mathrm{Hom}_{\mathrm{gr}\, A^\circ}(\mathbb{M}^\vee(1), {}_A\Bbbk(d+i)) = 0$. Hence $\underline{\mathrm{Ext}}^d_A(\Bbbk_A, \mathbb{M})$ is concentrated in degree $-d$. □

Theorem 4.6. *Retain the notation as above. We have*

(i) $\mathrm{End}_{\mathrm{gr}\, A}(\mathbb{M}) \cong C(A)$;

(ii) *A is a noncommutative isolated singularity if and only if* $\mathrm{End}_{\mathrm{gr}\, A}(\mathbb{M})$ *is semisimple.*

Proof. (i) From the resolution (4.2.10), we have the following exact triangle in $D^b(\operatorname{gr} A)$:

$$\Omega^d(\Bbbk_A)[d-1] \to P^{\cdot} \to \Bbbk_A \to \Omega^d(\Bbbk_A)[d],$$

where $P^{\cdot}$ is the complex $0 \to P^{-d+1} \xrightarrow{\partial^{-d+1}} \cdots \xrightarrow{\partial^1} P^0 \to 0$. Therefore

$$\Omega^d(\Bbbk_A)[d] \cong \Bbbk_A \tag{4.6.1}$$

in the quotient category $D^b(\operatorname{gr} A)/\operatorname{per} A$. Let F be the equivalence as in (4.2.9). Considering the equivalence functors (4.2.1)–(4.2.5), we have $F(\Omega^d(\Bbbk_A)[d]) = LB(\Omega^d(\Bbbk_A)[d]) = L\overline{K}G(\Omega^d(\Bbbk_A)[d])$. By (4.6.1), $G(\Omega^d(\Bbbk_A)[d]) \cong \Bbbk_A$. By (4.2.1) and (4.2.2), we have $F(\Omega^d(\Bbbk_A)[d]) \cong L\overline{K}(\Bbbk_A) \cong L(\pi(A^!))$. By (4.2.8), $L(\pi(A^!)) \cong C(A)$. Finally, we obtain that

$$F(\Omega^d(\Bbbk_A)[d]) \cong C(A). \tag{4.6.2}$$

Since F is an equivalence, we have

$$\operatorname{End}_{\underline{\mathrm{mcm}}\,A}(\Omega^d(\Bbbk_A)[d]) \cong \operatorname{End}_{D^b(\operatorname{mod} C(A))}(C(A)) \cong C(A),$$

whereas in the triangulated category $\underline{\mathrm{mcm}}\,A$, we have

$$\begin{aligned}\operatorname{End}_{\underline{\mathrm{mcm}}\,A}(\mathbb{M}) = \operatorname{End}_{\underline{\mathrm{mcm}}\,A}(\Omega^d(\Bbbk_A)(d)) &\cong \operatorname{End}_{\underline{\mathrm{mcm}}\,A}(\Omega^d(\Bbbk_A)) \\ &\cong \operatorname{End}_{\underline{\mathrm{mcm}}\,A}(\Omega^d(\Bbbk_A)[d]).\end{aligned}$$

Therefore,

$$\operatorname{End}_{\underline{\mathrm{mcm}}\,A}(\mathbb{M}) \cong C(A).$$

By Proposition 4.4, $\mathbb{M}^\vee(1)$ is a Koszul module. In particular, $\mathbb{M}^\vee$ is generated in degree 1 and $\operatorname{Hom}_{\operatorname{gr} A}(\mathbb{M}, A) = 0$, thus $\operatorname{End}_{\underline{\mathrm{mcm}}\,A}(\mathbb{M}) = \operatorname{End}_{\operatorname{gr} A}(M)$ and the desired isomorphism (i) follows.

The statement (ii) follows from [He and Ye 2019, Theorem 6.3] (see also [Mori and Ueyama 2019, Theorem 4.13]). □

Since A is a Koszul algebra, we may compute $\mathbb{M}$ and $\operatorname{End}_{\operatorname{gr} A}(\mathbb{M})$ by Koszul resolution of A. Now assume $A = T(V)/(R)$ for some finite-dimensional vector space V with generating relations $R \subseteq V \otimes V$. Let $C_0 = \Bbbk$, $C_1 = V$, $C_2 = R$ and $C_n = \bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j}$ for $n > 2$. Then the minimal projective resolution of $\Bbbk_A$ reads as follows (see [Beilinson et al. 1996]):

$$\cdots \to C_n \otimes A \xrightarrow{\partial^{-n}} C_{n-1} \otimes A \xrightarrow{\partial^{-(n-1)}} \cdots \xrightarrow{\partial^{-1}} C_0 \otimes A \to \Bbbk_A \to 0, \tag{4.6.3}$$

where the differential is defined as follows: if $\sum_{i=1}^m x_{1i} \otimes \cdots \otimes x_{ni} \in C_n$, for every $a \in A$,

$$\partial^{-n}\left(\left(\sum_{i=1}^m x_{1i} \otimes \cdots \otimes x_{ni}\right) \otimes a\right) = \sum_{i=1}^m (x_{1i} \otimes \cdots \otimes x_{n-1,i}) \otimes x_{ni}a.$$

By the above resolution, $\Omega^d(\Bbbk_A) \cong \operatorname{im} \partial^{-d}$. Consider the exact sequence

$$0 \to \operatorname{im} \partial^{-d-1} \to C_d \otimes A \to \Omega^d(\Bbbk_A) \to 0.$$

Since $C_d \otimes A$ is the projective cover of $\Omega^d(\Bbbk_A)$ and is generated in degree d, we get

$$\operatorname{End}_{\operatorname{gr} A}(\Omega^d(\Bbbk_A)) \cong \{f \in \operatorname{End}_{\operatorname{gr} A}(C_d \otimes A) \mid f(\operatorname{im} \partial^{-d-1}) \subseteq \operatorname{im} \partial^{-d-1}\}. \tag{4.6.4}$$

Notice that the restriction of ∂^{-d-1} on $(C_{d+1} \otimes A)_{d+1} \cong C_{d+1}$ is injective, and every element $f \in \operatorname{End}_{\operatorname{gr} A}(C_d \otimes A)$ is defined by its restriction on C_d. We thus have the isomorphism

$$\begin{aligned}\left\{f \in \operatorname{End}_{\operatorname{gr} A}(C_d \otimes A) \mid f(\operatorname{im} \partial^{-d-1}) \subseteq \operatorname{im} \partial^{-d-1}\right\}& \\ \cong \left\{f \in \operatorname{End}_{\Bbbk}(C_d) \mid (f \otimes 1)(C_{d+1}) \subseteq C_{d+1}\right\}&.\end{aligned} \tag{4.6.5}$$

Combining (4.6.4) and (4.6.5), we have the following isomorphism.

Proposition 4.7. *Write $A = T(V)/(R)$. We have*

$$\operatorname{End}_{\operatorname{gr} A}(\mathbb{M}) \cong \{f \in \operatorname{End}_{\Bbbk}(C_d) \mid (f \otimes 1)(C_{d+1}) \subseteq C_{d+1}\},$$

where $C_n = \bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j}$ for $n = d, d+1$, and $f \otimes 1$ is viewed as a linear map in $\operatorname{End}_{\Bbbk}(C_d \otimes V)$.

Remark 4.8. The proposition above provides a relatively easy way to compute the endomorphism ring of $\mathbb{M}$, especially when d is small. We will compute a detailed example of noncommutative quadric hypersurface of dimension 2 in the next section. According to Theorem 4.6 and the proposition above, to find whether A is a noncommutative isolated singularity is a linear algebra problem.

Let us check the MCM modules when A is a noncommutative isolated singularity.

Lemma 4.9. *Assume that A is a noncommutative isolated singularity. Each nonprojective indecomposable MCM A-module is isomorphic to a direct summand of $\mathbb{M}$ (up to a degree shift).*

Proof. By the isomorphism (4.6.2) in the proof of Theorem 4.6, $F(\Omega^d(\Bbbk_A)) \cong C(A)[-d]$. Since $C(A)$ is semisimple and F is an equivalence, all the indecomposable objects in $\underline{\operatorname{mcm}}\, A$ are direct summands of $\Omega^d(\Bbbk_A)$ (up to degree shifts). Notice that the class of nonprojective indecomposable MCM module over A is in one-to-one correspondence to the class of indecomposable objects in $\underline{\operatorname{mcm}}\, A$ (see [Smith and Van den Bergh 2013, Lemma 3.4]). Since $\mathbb{M}$ is a shift of $\Omega^d(\Bbbk_A)$, the result follows. □

We have the following properties of indecomposable MCM modules.

Proposition 4.10. *Retain the notation as above and keep Setup 4.2. Assume that A is a noncommutative isolated singularity.*

(i) *For $n \geq d$, $\dim \Omega^n(\Bbbk_A)_n = \frac{1}{2}\dim S^!$, where $\Omega^n(\Bbbk_A)_n$ is the degree n part of the graded module $\Omega^n(\Bbbk_A)$.*

(ii) $\dim \mathbb{M}_0 = \dim \operatorname{End}_{\operatorname{gr} A}(\mathbb{M})$.

(iii) *Suppose that $\operatorname{End}_{\operatorname{gr} A}(\mathbb{M})$ is a direct product of $\Bbbk$. If N is an indecomposable MCM module, then $N(i) \cong A/xA$ for some $i \in \mathbb{Z}$ and some element $x \in A_1$. Moreover, $\Omega(\mathbb{M}) \cong \mathbb{M}(-1)$.*

Proof. (i) By Lemma 4.1(ii), we have the exact sequence

$$0 \to A^!(-2) \xrightarrow{\cdot\varpi} A^! \to S^! \to 0.$$

Then we have $\dim A^!_i = \dim A^!_{i-2} + \dim S^!_i$ for all $i \geq 2$, and $\dim A^!_i = \dim S^!_i$ for $i = 0, 1$. Then by an iterative computation, we have

$$\dim A^!_n = \begin{cases} \dim S^!_0 + \dim S^!_2 + \cdots + S^!_n & \text{when } n \text{ is even;} \\ \dim S^!_1 + \dim S^!_3 + \cdots + S^!_n & \text{when } n \text{ is odd.} \end{cases}$$

Since S is a quantum polynomial algebra, $H_{S^!}(t) = (1+t)^{d+1}$. Therefore $\dim A^!_n = \frac{1}{2}\dim S^!$ for $n \geq d$. Since A is a Koszul algebra, we have

$$\dim A^!_n = \dim \operatorname{Hom}_{\operatorname{gr} A}(\Omega^n(\Bbbk_A)(n), \Bbbk_A) = \dim \Omega^n(\Bbbk_A)_n,$$

and therefore $\dim \Omega^n(\Bbbk_A)_n = \frac{1}{2}\dim S^!$.

(ii) By Theorem 4.6, $\operatorname{End}_{\operatorname{gr} A}(\mathbb{M}) \cong C(A)$. Hence $\dim \operatorname{End}_{\operatorname{gr} A}(\mathbb{M}) = \dim C(A) = \frac{1}{2}\dim S^!$ (see (4.2.7)). Since $\mathbb{M} = \Omega^d(\Bbbk_A)(d)$, the identity follows from (i).

(iii) By Lemma 4.9, it suffices to show that the result holds for each indecomposable direct summand N of $\mathbb{M}$.

Assume $\mathbb{M} = \mathbb{M}^1 \oplus \cdots \oplus \mathbb{M}^s$, where each $\mathbb{M}^i$ is indecomposable. By assumption $\operatorname{End}_{\operatorname{gr} A}(\mathbb{M})$ is a direct product of $\Bbbk$, which forces that $\operatorname{End}_{\operatorname{gr} A}(\mathbb{M}^i) = \Bbbk$ for all i and $\operatorname{Hom}_{\operatorname{gr} A}(\mathbb{M}^i, \mathbb{M}^j) = 0$ for $i \neq j$. Hence $s = \dim \operatorname{End}_{\operatorname{gr} A}(\mathbb{M})$, and $\mathbb{M}$ has no projective direct summands, otherwise if some $\mathbb{M}^i$ is projective, then $\operatorname{Hom}_{\operatorname{gr} A}(\mathbb{M}^i, \mathbb{M}^j) \neq 0$ for any j. Since $\mathbb{M}$ is a Koszul module, each $\mathbb{M}^i$ is a Koszul module. Hence $\mathbb{M}^i$ is generated in degree 0 for every i. By (i) and (ii), $s = \dim \mathbb{M}_0 = \frac{1}{2}\dim S^!$. Therefore, each $\mathbb{M}^i$ is a cyclic module. Hence we have the exact sequence

$$0 \to \Omega(\mathbb{M}^i) \to A \to \mathbb{M}^i \to 0 \quad \text{for all } 1 \leq i \leq s.$$

Then $\Omega^{d+1}(\Bbbk_A)(d) \cong \Omega(\mathbb{M}) \cong \bigoplus_{i=1}^s \Omega(\mathbb{M}^i)$. By (i), $\dim \Omega^{d+1}(\Bbbk_A)_{d+1} = s$. Hence $\dim \Omega(\mathbb{M}^i)_1 = 1$ for all $1 \leq i \leq s$ since each $\mathbb{M}^i$ is not projective. Note that $\Omega(\mathbb{M}^i)$ is generated in degree 1. It follows that there is an element $x_i \in A_1$ such that $\Omega(\mathbb{M}^i) \cong x_i A$ for all $1 \leq i \leq s$. Hence $\mathbb{M}^i \cong A/x_i A$.

Since $\operatorname{Hom}_{\operatorname{gr} A}(\mathbb{M}^i, \mathbb{M}^j) = 0$ for $i \neq j$, we have $\mathbb{M}^i \ncong \mathbb{M}^j$ if $i \neq j$. Therefore $\Omega(\mathbb{M}^i) \ncong \Omega(\mathbb{M}^j)$ for $i \neq j$ since $\Omega = [-1]$ is the suspension functor in the triangulated category $\underline{\mathrm{mcm}}\, A$. Since each $\Omega(\mathbb{M}^i)$ is indecomposable and generated in degree 1, it follows that the set $\{\Omega(\mathbb{M}^1), \dots, \Omega(\mathbb{M}^s)\} = \{\mathbb{M}^1(-1), \dots, \mathbb{M}^s(-1)\}$ by Lemma 4.9. Therefore $\Omega(\mathbb{M}) \cong \mathbb{M}(-1)$. □

Lemma 4.9 also provides a way to construct noncommutative resolution of noncommutative isolated singularities.

Theorem 4.11. *Keep the notions in Setup 4.2. Assume that A is a noncommutative isolated singularity. Then $B = \underline{\mathrm{End}}_A(\mathbb{M} \oplus A)$ is a right preresolution of A. Moreover, B is concentrated in nonnegative degrees.*

Proof. That B is a right preresolution of A follows from Theorem 3.6 and Lemma 4.9. Since $\mathbb{M}$ is a Koszul module, B is concentrated in nonnegative degrees. □

Remark 4.12. The algebra B is isomorphic to the matrix algebra $\begin{pmatrix} \underline{\mathrm{End}}_A(\mathbb{M}) & \mathbb{M} \\ \mathbb{M}^\vee & A \end{pmatrix}$. By Proposition 4.4, $\mathbb{M}^\vee$ is concentrated in degrees not less than 1. By Proposition 4.5(ii), every element $f \in \underline{\mathrm{End}}_A(\mathbb{M})_{\geq 1}$ factors through a projective module. By [McConnell and Robson 1987, Proposition 7.5.1],

$$B_0 = \begin{pmatrix} \operatorname{End}_{\operatorname{gr} A}(\mathbb{M}) & \mathbb{M}_0 \\ 0 & \Bbbk \end{pmatrix}$$

is of global dimension 1, since $\operatorname{End}_{\operatorname{gr} A}(\mathbb{M})$ is semisimple by assumption. In the next section, we give a concrete example with detailed computations of elements of B.

5. An example

In this section, we give a detailed computation of indecomposable MCM module of an explicit noncommutative quadric hypersurfaces. Let $\Bbbk = \mathbb{C}$, let $S = \Bbbk\langle x, y, z\rangle/(R)$, where $R = \operatorname{span}\{xz + zx, yz + zy, x^2 + y^2\}$. Then S is a quantum polynomial algebra of global dimension 3, which is an AS-regular algebra of type S_2 as listed in [Artin and Schelter 1987, Table 3.11, p. 183].

The following facts were proved in [He and Ye 2019, Setion 9]; see also [Hu 2022] for a complete classification of noncommutative conics.

Lemma 5.1. *Let $\varpi = x^2 + z^2 \in S_2$. Then*

(i) *ϖ is a central regular element of S,*

(ii) *$A = S/S\varpi$ is a noncommutative isolated singularity,*

(iii) *$C(A) \cong \Bbbk^4$, where $C(A)$ is the algebra defined in (4.2.6).*

Let $V = \operatorname{span}\{x, y, z\}$, and write $A = T(V)/(R')$, where R' is spanned by

$$x \otimes z + z \otimes x, \quad y \otimes z + z \otimes y, \quad x \otimes x + y \otimes y, \quad x \otimes x + z \otimes z.$$

Consider the Koszul resolution of $\Bbbk_A$:

$$\cdots \to ((R' \otimes V) \cap (V \otimes R')) \otimes A \xrightarrow{\partial^{-3}} R' \otimes A \xrightarrow{\partial^{-2}} V \otimes A \xrightarrow{\partial^{-1}} A \to \Bbbk_A \to 0.$$

By a direct calculation, $(R' \otimes V) \cap (V \otimes R')$ has a basis

$$(x \otimes z + z \otimes x) \otimes x + (y \otimes z + z \otimes y) \otimes y + (x \otimes x + y \otimes y) \otimes z,$$

$$\begin{aligned} 2(x \otimes z + z \otimes x) \otimes x &+ (y \otimes z + z \otimes y) \otimes y \\ &+ (x \otimes x + y \otimes y) \otimes z + (x \otimes x + z \otimes z) \otimes z, \end{aligned}$$

$$(y \otimes z + z \otimes y) \otimes z - (x \otimes x + y \otimes y) \otimes y + (x \otimes x + z \otimes z) \otimes y,$$

$$\begin{aligned} (x \otimes z + z \otimes x) \otimes z &+ (y \otimes z + z \otimes y) \otimes z \\ &- (x \otimes x + y \otimes y) \otimes y + (x \otimes x + z \otimes z) \otimes (x + y). \end{aligned}$$

Set $\mathbb{M} = \Omega^2(\Bbbk_A)(2) = \operatorname{im} \partial^{-2}(2)$. Then $\mathbb{M}$ is a Koszul module which is generated by $\frac{1}{2} \dim S^!(=4)$ elements, see Proposition 4.10(i). The set of relations between the generators of $\mathbb{M}$ is equal to $\operatorname{im} \partial^{-3}(2)$ which is also generated by 4 elements. Thus we may write $\mathbb{M}$ as a quotient module of a free module:

$$0 \to K \to m_1 A \oplus m_2 A \oplus m_3 A \oplus m_4 A \to \mathbb{M} \to 0,$$

where $\{m_1, m_2, m_3, m_4\}$ is a free basis, and K is the submodule generated by

$$\begin{aligned} r_1 &= m_1 x + m_2 y + m_3 z, \\ r_2 &= 2m_1 x + m_2 y + m_3 z + m_4 z, \\ r_3 &= m_2 z - m_3 y + m_4 y, \\ r_4 &= m_1 z + m_2 z - m_3 y + m_4 (x + y). \end{aligned}$$

To find indecomposable MCM A-modules, we only need to find a set of primitive idempotents of $\operatorname{End}_{\operatorname{gr} A}(\mathbb{M})$ by Lemma 4.9. Let $F = m_1 A \oplus m_2 A \oplus m_3 A \oplus m_4 A$. Note that the degree one part of K is $K_1 = \operatorname{span}\{r_1, r_2, r_3, r_4\}$. We have

$$\operatorname{End}_{\operatorname{gr} A}(\mathbb{M}) = \{\theta \in \operatorname{End}_{\operatorname{gr} A}(F) \mid \theta(r_j) \in K_1,\ j = 1, 2, 3, 4\}.$$

By some computations on linear equations, we have

$$\operatorname{End}_{\operatorname{gr} A}(\mathbb{M}) = \left\{ \begin{pmatrix} b+d & 0 & a & a \\ 0 & b & c & 0 \\ 0 & -c & b & 0 \\ a & c & d & b+d \end{pmatrix} : a, b, c, d \in \Bbbk \right\}.$$

We have the following complete set of primitive idempotents in $\mathrm{End}_{\mathrm{gr}\,A}(\mathbb{M})$:

$$e_1=\begin{pmatrix}0&0&0&0\\0&\frac12&\frac12 i&0\\0&-\frac12 i&\frac12&0\\0&\frac12 i&-\frac12&0\end{pmatrix},\qquad e_2=\begin{pmatrix}0&0&0&0\\0&\frac12&-\frac12 i&0\\0&\frac12 i&\frac12&0\\0&-\frac12 i&-\frac12&0\end{pmatrix},$$

$$e_3=\begin{pmatrix}\frac12&0&\frac12&\frac12\\0&0&0&0\\0&0&0&0\\\frac12&0&\frac12&\frac12\end{pmatrix},\qquad e_4=\begin{pmatrix}\frac12&0&-\frac12&-\frac12\\0&0&0&0\\0&0&0&0\\-\frac12&0&\frac12&\frac12\end{pmatrix},$$

where $i=\sqrt{-1}$ is a square root of -1. Therefore, we have the following nonprojective nonisomorphic indecomposable MCM modules:

$$\begin{aligned}\mathbb{M}^1&=\mathbb{M}e_1=(\overline{m}_2-i\overline{m}_3+i\overline{m}_4)A,\\ \mathbb{M}^2&=\mathbb{M}e_2=(\overline{m}_2+i\overline{m}_3-i\overline{m}_4)A,\\ \mathbb{M}^3&=\mathbb{M}e_3=(\overline{m}_1+\overline{m}_4)A,\\ \mathbb{M}^4&=\mathbb{M}e_4=(\overline{m}_1-\overline{m}_4)A,\end{aligned}$$

where $\overline{m}_1,\overline{m}_2,\overline{m}_3,\overline{m}_4$ are the image of m_1,m_2,m_3,m_4 in $\mathbb{M}$.

Since $\mathbb{M}^1,\dots,\mathbb{M}^4$ are Koszul modules, a straightforward check shows that

$$\mathbb{M}^1\cong A/(y+iz)A,\quad \mathbb{M}^2\cong A/(y-iz)A,\quad \mathbb{M}^3\cong A/(x+z)A,\quad \mathbb{M}^4\cong A/(x-z)A.$$

Note that $y+iz$, $y-iz$, $x+z$, $x-z$ are nilpotent elements in A. Also, we have

$$\begin{aligned}A/(y+iz)A\cong(y+iz)A(1),&\quad A/(y-iz)A\cong(y-iz)A(1),\\ A/(x+z)A\cong(x+z)A(1),&\quad A/(x-z)A\cong(x-z)A(1).\end{aligned}$$

Summarizing, we have the following conclusion.

Proposition 5.2. *The quadric hypersurface A has nonprojective nonisomorphic indecomposable MCM modules* (*up to degree shifts*):

$$\begin{aligned}\mathbb{M}^1&\cong A/(y+iz)A\cong(y+iz)A(1),\\ \mathbb{M}^2&\cong A/(y-iz)A\cong(y-iz)A(1),\\ \mathbb{M}^3&\cong A/(x+z)A\cong(x+z)A(1),\\ \mathbb{M}^4&\cong A/(x-z)A\cong(x-z)A(1).\end{aligned}$$

By Theorem 4.11, A has a right preresolution $\underline{\mathrm{End}}_A(\mathbb{M}\oplus A)$. Write $u_1=y+iz$, $u_2=y-iz$, $u_3=x+z$, $u_4=x-z$. Note that A is an AS-Gorenstein algebra (see Lemma 4.1(ii)). Since A/u_iA is an MCM module, we have $\underline{\mathrm{Ext}}^1_A(A/u_iA,A)=0$ for each $1\le i\le 4$. Hence the map $\tau^\vee:\underline{\mathrm{Hom}}_A(A,A)\to\underline{\mathrm{Hom}}_A(u_iA,A)$ induced from the inclusion map $\tau : u_i A \to A$ is surjective. Therefore, for each homogeneous element $f \in \underline{\mathrm{Hom}}_A(u_i A, A)$, there is a homogeneous element $a \in A$ such that $f(u_i) = au_i$. On the other hand, for each homogeneous element $a \in A$, there is a graded right A-module morphism $f : u_i A \to A$ defined by $f(u_i) = au_i$. Hence, we obtain $\underline{\mathrm{Hom}}_A(u_i A, A) \cong Au_i(1)$.

Since A is AS-Gorenstein, $\underline{\mathrm{mcm}}\, A$ is semisimple and $C(A) \cong \mathrm{End}_A(\mathbb{M}) \cong \Bbbk^4$, by the equivalence functor (4.2.9), we have $\underline{\mathrm{Ext}}^1_A(u_i A, u_j A) = 0$ for all $i \neq j$ and $\underline{\mathrm{Ext}}^1_A(u_i A, u_i A) \cong \Bbbk$. Therefore, the exact sequence $0 \to u_i A \to A \to A/u_i A \to 0$ induces a surjective map $\underline{\mathrm{Hom}}_A(A, u_j A) \to \underline{\mathrm{Hom}}_A(u_i A, u_j A)$ for $i \neq j$. Then we get $\underline{\mathrm{Hom}}_A(u_i A, u_j A) = u_j A u_i(1)$ for $i \neq j$. Similarly, $\underline{\mathrm{End}}_A(u_i A) = \Bbbk \oplus u_i A u_i(1)$ for $i = 1, 2, 3, 4$. Then $\underline{\mathrm{End}}_A(\mathbb{M} \oplus A)$ is isomorphic to the algebra

$$\begin{pmatrix} u_1Au_1(1) & u_1Au_2(1) & u_1Au_3(1) & u_1Au_4(1) & u_1A(1) \\ u_2Au_1(1) & u_2Au_2(1) & u_2Au_3(1) & u_2Au_4(1) & u_2A(1) \\ u_3Au_1(1) & u_3Au_2(1) & u_3Au_3(1) & u_3Au_4(1) & u_3A(1) \\ u_4Au_1(1) & u_4Au_2(1) & u_4Au_3(1) & u_4Au_4(1) & u_4A(1) \\ Au_1 & Au_2 & Au_3 & Au_4 & A_{\geq 1} \end{pmatrix} \bigoplus \begin{pmatrix} \Bbbk & 0 & 0 & 0 & 0 \\ 0 & \Bbbk & 0 & 0 & 0 \\ 0 & 0 & \Bbbk & 0 & 0 \\ 0 & 0 & 0 & \Bbbk & 0 \\ 0 & 0 & 0 & 0 & \Bbbk \end{pmatrix},$$

where the multiplication is defined as below. For consistency of notations we set $u_5 = 1$. We simply write elements in the left matrix as $(u_i a_{ij} u_j)$. Then

$$(u_i a_{ij} u_j)(u_i b_{ij} u_j) = \left(\sum_{k=1}^{5} u_i a_{ik} u_k b_{kj} u_j \right).$$

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